If it is raining, then I will bring an umbrella ⟸ If I bring an umbrella, then it is raining

If it is not raining, then I won't bring an umbrella

5.3: Divergence and Integral Tests

Math 253, Spring 2021

PQ. Recall the following fact that you proved on the first homework. If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Which of the following statements is equivalent to this fact?

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ converges

(b) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

(c) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$.

$\sum_{n=1}^{\infty} \frac{1}{n}$

a_0
$a_0 + a_1$
$a_0 + a_1 + a_2$
$\vdots$
$a_0 + a_1 + \cdots + a_k$
$\downarrow$
L

$\Rightarrow$

a_0
a_1
a_2
a_3
$\vdots$
$\downarrow$
0

Thm: (The Divergence Test) If the sequence of terms in a series do not converge to 0, then the series diverges.

Q. If you are asked to determine whether a series converges or diverges, when should you use this test?

First: the test is fast!

Ex 1 Does the series $\sum_{n=1}^{\infty} \cos(1/n)$ converge or diverge?

$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right) = \cos(0) = 1 \neq 0$

$\to \sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ diverges

Ex 2 Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

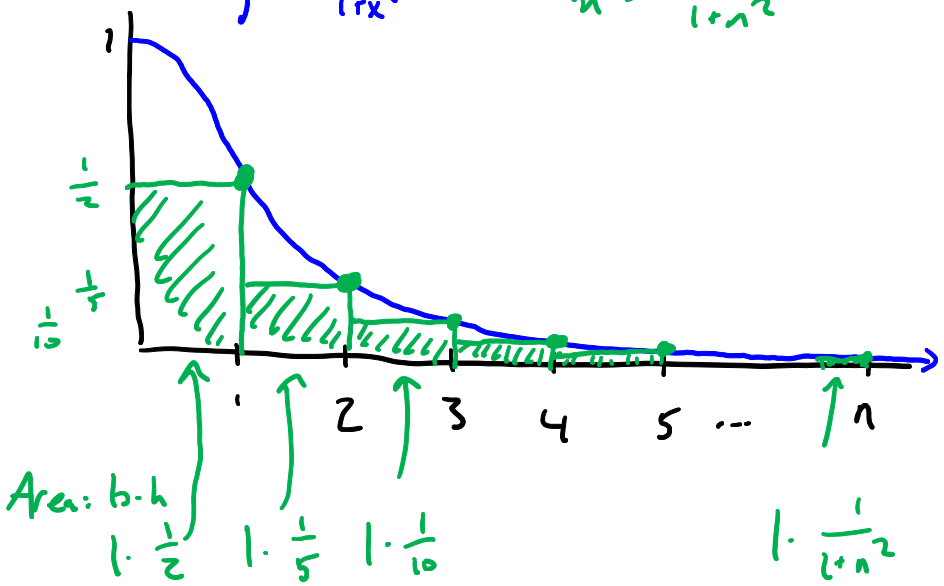
$\lim_{n\to\infty} \frac{1}{n} = 0$

$\to$ no conclusion

Note (IMPORTANT) If $\lim_{n\to\infty} a_n = 0$, the divergence test tells you nothing.

Q. How can we relate series to things that we already understand? Like the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$?

$y = \frac{1}{1+x^2}$ $\quad a_n = \frac{1}{1+n^2}$ for $n \geq 1$



Total green area: $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots + \frac{1}{1+n^2} + \cdots$

$$\leq \int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \arctan(x)\Big|_0^t$$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(0)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

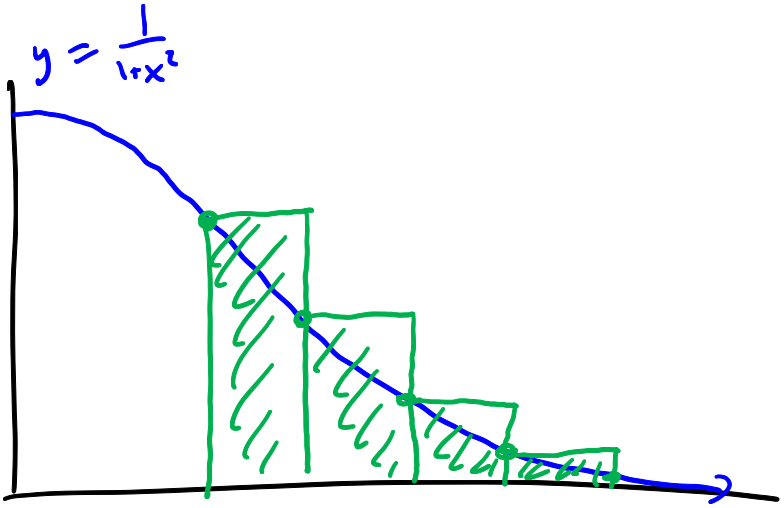
$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \leq \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} \longrightarrow S_k$ is bded.

$S_k = \frac{1}{1+1^2} + \frac{1}{1+2^2} + \cdots \frac{1}{1+k^2} \leq \frac{1}{1+1^2} + \cdots + \frac{1}{1+k^2} + \frac{1}{1+(k+1)^2} = S_{k+1}$

$\longrightarrow S_k$ is monotone $\Rightarrow S_k$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges

Q. What is $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$?

$\longrightarrow \quad \leq \frac{\pi}{2}$



$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ = green area $\leq \int_1^{\infty} \frac{1}{1+x^2}\,dx$

$= \frac{\pi}{2} - \arctan(1)$

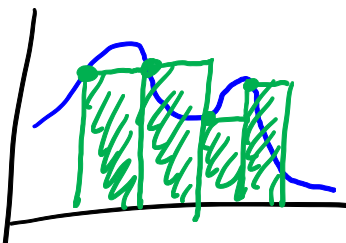
$= \frac{\pi}{4}$

$$\frac{\pi}{4} \leq \sum_{n=1}^{\infty} \frac{1}{1+n^2} \leq \frac{\pi}{2}$$

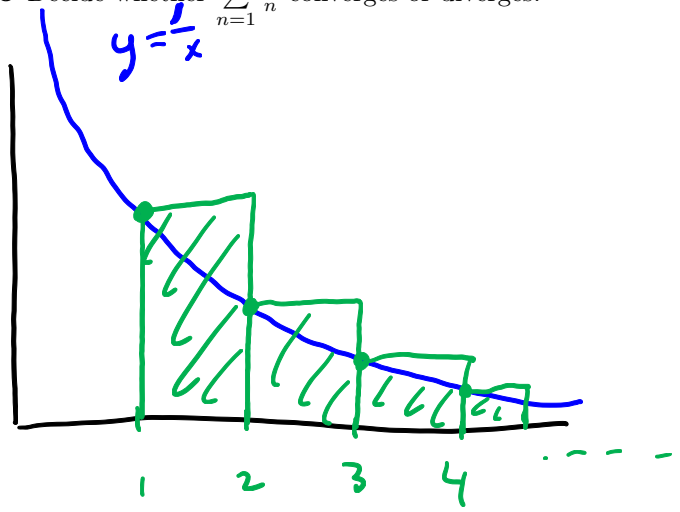
Note: We are using

① $\frac{1}{1+x^2} \geq 0 \longrightarrow$ use MCT

② $\frac{1}{1+x^2}$ is monotone decreasing (in eqs.)



Ex 3 Decide whether $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.



$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \geq \int_1^{\infty} \frac{1}{x}\,dx$$

$$\lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

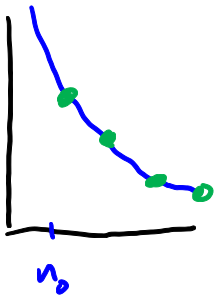
$$\lim_{t\to\infty} \ln|x| \Big|_1^t$$

$$\lim_{t\to\infty} \ln|t| - \ln(1)$$

$$\infty$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞

Thm: (The Integral Test) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms. Suppose there is a function f and a positive integer n_0 such that the following conditions are met:



Ex: $a_n = \frac{1}{1+n^2}$ → $f(x) = \frac{1}{1+x^2}$



(a) f is continuous for all $x \geqslant n_0$

(b) f is decreasing for all $x \geqslant n_0$

(c) $f(n) = a_n$ for all integers $n \geqslant n_0$ (ie. f interpolates $\{a_n\}$ for all $n \geqslant n_0$)

Nonex: $a_n = (-1)^n \frac{1}{n}$

$f(x) = (-1)^x \frac{1}{x}$ ↑ not defined for non-integer x

Then

- If $\int_{n_0}^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\int_{n_0}^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Q. What is great about this theorem?

- Gives a test for convergence and divergence simultaneously

Q. What is not great about this theorem?

- Integrals are hard!
- $f(x)$ doesn't always exist

Def: If p is a real number, we call the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ a p-series.

Q. For which values of p does the p-series converge?

$p = 1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$p < 0$

(think: $p = -1$)

$\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$

diverges by the div. test (since $\frac{1}{n^p} \xrightarrow[n\to\infty]{} \infty$)

$p = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$

diverge by the div. test (since $\frac{1}{n^p} \xrightarrow[n\to\infty]{} 1$)

$p > 0,\ p \neq 1$

(think: $p = \frac{1}{2}$)

$\sum_{n=1}^{\infty} \frac{1}{n^p}$

$f(x) = \frac{1}{x^p}$

cts. ✓

dec. ✓

$f(n) = \frac{1}{n^p}$ ✓

$f(x) > 0$ ✓



$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges

$$\lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t$$

>0 when $p<1$

<0 when $p>1$

$$= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}$$

$$= \begin{cases} \infty & p<1 \\ -\frac{1}{1-p} = \frac{1}{p-1} & p>1 \end{cases}$$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p>1$

Ex 4 Decide whether the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$ converges or not.

(a) Converges
(b) Diverges
(c) Neither

$\frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \frac{1}{4^{1/2}} + \cdots$ ← $p = \frac{1}{2}$

Q. We know that if $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = L$ for *some* L. Can we find the value of L?

No: $S_k = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$ ← can't simplify

Fact: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$

Thm: (Integral Test Remainder Estimate) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms. Suppose there is a function f and a positive integer k such that the following conditions are met:

(a) f is continuous for $x \geqslant k$
(b) f is decreasing for $x \geqslant k$
(c) $f(n) = a_n$ for all integers $n \geqslant k$.

Then if $\sum_{n=1}^{\infty} a_n$ converges, we get a remainder estimate:

$$\int_{k+1}^{\infty} f(x)\,dx \leq R_k \leq \int_{k}^{\infty} f(x)\,dx$$

If $\sum_{n=1}^{\infty} a_n = L$, then $S_k \approx L$

$S_k = a_1 + a_2 + \cdots + a_k$

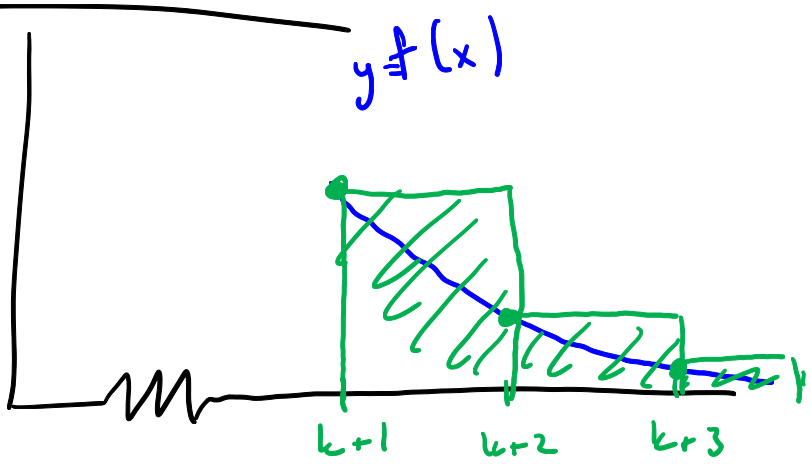
$\rightarrow L - S_k \approx 0$

$\rightarrow (a_1 + a_2 + a_3 + \cdots) - (a_1 + a_2 + \cdots + a_k) \approx 0$

$a_{k+1} + a_{k+2} + \cdots \approx 0$

$R_k = \sum_{n=k+1}^{\infty} a_n$ ← remainder error

Lower Bound



$$\int_{k+1}^{\infty} f(x)\,dx \le R_k = a_{k+1} + a_{k+2} + \cdots$$

Upper Bound



$$R_k = a_{k+1} + a_{k+2} + \cdots \le \int_k^{\infty} f(x)\,dx$$

Ex 5 Consider $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$.

(a) Find explicit upper and lower bounds for $R_k = \sum_{n=k+1}^{\infty} \frac{1}{1+n^2}$ in terms of k.

$$\int_{k+1}^{\infty} \frac{1}{1+x^2}\,dx \le R_k \le \int_{k}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\frac{\pi}{2} - \arctan(k+1) \le R_k \le \frac{\pi}{2} - \arctan(k)$$

(b) Use the partial sum $S_{1000} = \sum_{n=1}^{1000} \frac{1}{1+n^2}$ to estimate the value of the "full" series $L = \sum_{n=1}^{\infty} \frac{1}{1+n^2}$, and give a bound for the error.

Computer: $S_{1000} \approx 1.076$

Error: $R_{1000} \le \frac{\pi}{2} - \arctan(1000) \approx 0.001$

Conclude: $1.075 \le \sum_{n=1}^{\infty} \frac{1}{1+n^2} \le 1.077$

(c) Repeat (b) with S_{100000}.

Computer: $S_{100\,000} \approx 1.0766040$

Error: $R_{100\,000} \le \frac{\pi}{2} - \arctan(100\,000) \approx 0.0001$

Conclude: $1.0765040 \le \sum_{n=1}^{\infty} \frac{1}{1+n^2} \le 1.0767040$

Exact: $\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1 + e^{2\pi}(\pi - 1) - \pi}{2(e^{2\pi} - 1)}$

Ex 6 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$.

(a) Why does the series converge?

p-series with $p = 5 > 1 \rightarrow \sum_{n=1}^{\infty} \frac{1}{n^5}$ converges

(b) Estimate L with error $< 10^{-5}$.

$L \approx S_k$

Goal: Find k with $R_k < 10^{-5}$

Since $R_k \leq \int_k^{\infty} \frac{1}{x^5}\,dx$, if we find k s.t.

$\int_k^{\infty} \frac{1}{x^5}\,dx < 10^{-5}$, we're done!

Goal: $\int_k^{\infty} \frac{1}{x^5}\,dx < 10^{-5}$

LHS: $\int_k^{\infty} \frac{1}{x^5}\,dx = \lim_{t\to\infty} \int_k^t x^{-5}\,dx$

$= \lim_{t\to\infty} \frac{x^{-4}}{-4}\Big|_k^t$

$= \lim_{t\to\infty} \frac{t^{-4}}{-4} - \frac{k^{-4}}{-4} = \frac{1}{4k^4}$

Goal: $\frac{1}{4k^4} < 10^{-5} \leftrightarrow \frac{10^5}{4} < k^4 \leftrightarrow \left(\frac{10^5}{4}\right)^{1/4} < k$

≈ 12.57

$\rightarrow k = 13$

Know: $R_{13} < 10^{-5}$

Conclude: $S_{13} = \sum_{n=1}^{13} \frac{1}{n^5} \approx 1.03692$

with error < 0.00001